1. Use the differential area $\mathbf{dS}$ to calculate the area of the surface defined by $\rho = 5$, $\pi/2 < \phi > \pi$, and $-2 < z < 2$. [5]

Solution. $\mathrm{d}\rho = 0$, $\mathrm{d}\mathbf{S} = \rho\,\mathrm{d}\phi\,\mathrm{d}z\mathbf{a}_\rho$;

$$\begin{aligned} S &= \int_{-2}^{2}\int_{\pi/2}^{\pi} \rho\,\mathrm{d}\phi\,\mathrm{d}z \\ &= 5(2+2)\left(\pi - \frac{\pi}{2}\right) \\ &= 5(4)\frac{\pi}{2} \\ &= 10\pi \end{aligned}$$

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2. Convert the vector $\mathbf{P} = y^2\mathbf{p}_x + (x+1)\mathbf{p}_y + yz\mathbf{p}_z$ into cylindrical coordinates. [5]

Solution.

$$\begin{aligned}
\begin{bmatrix} p_\rho \\ p_\phi \\ p_z \end{bmatrix} &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \\
&= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^2 \\ x+1 \\ yz \end{bmatrix}
\end{aligned}$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$ and $z = z$.

$$\begin{bmatrix} p_\rho \\ p_\phi \\ p_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \begin{bmatrix} \rho^2\sin^2\phi \\ \rho\cos\phi + 1 \\ \rho z\sin\phi \end{bmatrix}$$

$$\mathbf{P} = \left(\rho^2\sin^2\phi\cos\phi + \sin\phi(\rho\cos\phi + 1)\right)\mathbf{p}_\rho + \left(-\rho^2\sin^3\phi + \cos\phi(\rho\cos\phi + 1)\right)\mathbf{p}_\phi + \rho z\sin\phi\mathbf{p}_z$$

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3. Given two points, namely $A(2, 60°, 2)$ and $B(2\sqrt{3}, 30°, 3)$, find the corresponding position vectors **A** and respectively **B**. Then find the unit vector in the direction from **A** to **B**.

Solution. $x = \rho \cos\phi$, $y = \rho \sin\phi$ and $z = z$; $\mathbf{A}(2\cos 60°, 2\sin 60°, 2) \to \mathbf{A}(1, \sqrt{3}, 2)$

$$\mathbf{B}(2\sqrt{3}\cos 30°, 2\sqrt{3}\sin 30°, 3) \to \mathbf{B}(3, \sqrt{3}, 3)$$

$$\begin{aligned} \mathbf{B} - \mathbf{A} &= (3, \sqrt{3}, 3) - (1, \sqrt{3}, 2) = (2, 0, 1) \\ |\mathbf{B} - \mathbf{A}| &= \sqrt{4+1} = \sqrt{5} \\ \mathbf{a} &= \frac{(2,0,1)}{\sqrt{5}} = \frac{2}{\sqrt{5}}\mathbf{a}_x + \frac{1}{\sqrt{5}}\mathbf{a}_z \end{aligned}$$

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4. Find $\nabla^2\mathbf{B}$ when $\mathbf{B} = r^2\mathbf{b}_r + \sin\theta\mathbf{b}_\theta + \cos^2\theta\mathbf{b}_\phi$.

Solution.

$$\begin{aligned}\nabla\cdot\mathbf{B} &= \frac{1}{r^2}(4r^3) + \frac{1}{\sin\theta}(2\sin\theta\cos\theta)\\ &= 4r + \frac{2}{r}\cos\theta\end{aligned}$$

$$\nabla(\nabla\cdot\mathbf{B}) = \left(4 - \frac{2}{r^2}\cos\theta\right)\mathbf{a}_r - \frac{2}{r^2}\sin\theta\mathbf{a}_\theta$$

$$\nabla\times\mathbf{B} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\cos^2\theta\sin\theta)\mathbf{a}_r + \frac{1}{r}[-\cos^2\theta]\mathbf{a}_\theta + \frac{1}{r}(\sin\theta)\mathbf{a}_\phi$$

$\frac{\partial}{\partial\theta}(\cos^2\theta\sin\theta) = \frac{\partial}{\partial\theta}(\sin\theta - \sin^3\theta) = \cos\theta - 3\sin^2\theta\cos\theta$;

$$\nabla\times\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial\theta}(\cot\theta - 3\sin\theta\cos\theta)\mathbf{a}_r - \frac{1}{r}\cos^2\theta\mathbf{a}_\theta + \frac{1}{r}\sin\theta\mathbf{a}_\phi$$

But $3\sin\theta\cos\theta = \frac{3}{2}\sin 2\theta$, therefore $\frac{3}{2}\frac{\partial}{\partial\theta}\sin 2\theta = 3\cos 2\theta$.

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{B}) &= \frac{1}{\sin\theta}\left[\frac{1}{r}(2\sin\theta\cos\theta)\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{r}(\csc^2\theta + 3\cos 2\theta)\right]\mathbf{a}_\phi\\ &= \frac{2}{r^2}\cos\theta\mathbf{a}_r + \frac{1}{r^2}(\csc^2\theta + 3\cos 2\theta)\mathbf{a}_\phi\end{aligned}$$

Therefore

$$\nabla^2\mathbf{B} = \left(4 - \frac{4}{r^2}\cos\theta\right)\mathbf{a}_r - \frac{2}{r^2}\sin\theta\mathbf{a}_\theta - \frac{1}{r^2}(\csc^2\theta + 3\cos 2\theta)\mathbf{a}_\phi.$$

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